Rational dynamics on the projective line of the field of p-adic numbers

Lingmin LIAO (Université Paris-Est Créteil)

(joint with **Shi-Lei Fan**, **Hongming Nie**, **Yue-Fei Wang**)

Eighth International Conference on p-adic Mathematical Physics and its Applications

Web conference, May 25th 2021

Outline

1. Introduction
2. 1-Lipschitz dynamical systems on $\mathbb{Q}_p$
3. Expanding dynamical systems on $\mathbb{Q}_p$
4. Geometrically finite rational dynamics
5. An example : $x \mapsto \frac{9}{4}x(x-1)^2$ on $\mathbb{P}^1(\mathbb{Q}_2)$

Introduction

I. Projective line over $\mathbb{Q}_p$

For $(x_1, y_1), (x_2, y_2) \in \mathbb{Q}_p^2 \setminus \{(0,0)\}$, we say that $(x_1, y_1) \sim (x_2, y_2)$ if

$$\exists \lambda \in \mathbb{Q}_p^* \ \text{ s.t. } \quad x_1 = \lambda x_2 \text{ and } y_1 = \lambda y_2.$$

Projective line over $\mathbb{Q}_p$:

$$\mathbb{P}^1(\mathbb{Q}_p) := (\mathbb{Q}_p^2 \setminus \{(0,0)\}) / \sim .$$

Spherical metric : for $P = [x_1, y_1], Q = [x_2, y_2] \in \mathbb{P}^1(\mathbb{Q}_p)$, define

$$\rho(P, Q) = \frac{|x_1 y_2 - x_2 y_1|_p}{\max\{|x_1|_p, |y_1|_p\} \max\{|x_2|_p, |y_2|_p\}}.$$

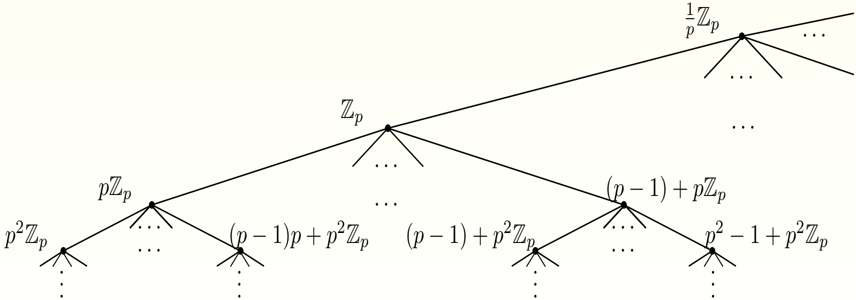
Viewing $\mathbb{P}^1(\mathbb{Q}_p)$ as $\mathbb{Q}_p \cup \{\infty\}$, for $z_1, z_2 \in \mathbb{Q}_p \cup \{\infty\}$ we define

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}} \qquad \text{if } z_1, z_2 \in \mathbb{Q}_p,$$

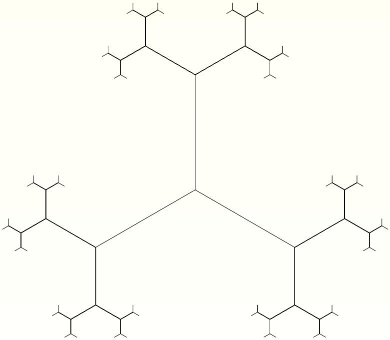
and

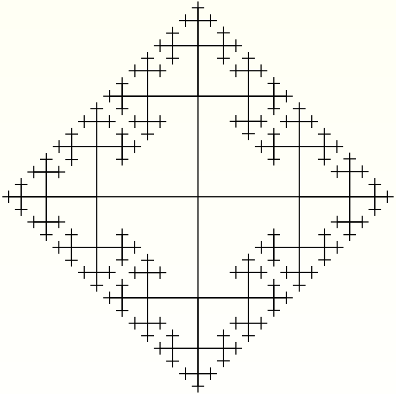
$$\rho(z, \infty) = \begin{cases} 1, & \text{if } |z|_p \le 1\,; \\ 1/|z|_p, & \text{if } |z|_p > 1. \end{cases}$$

Tree representation of $\mathbb{Q}_p$



Bruhat–Tits trees :





II. p-adic dynamical systems

A p-adic dynamical system is a couple (X, f) where X is a p-adic space and $f : X \to X$ is a transformation on X.

The beginning :

- Oselies-Zieschang 1975 : automorphisms of $\mathbb{Z}_p$,
- Herman-Yoccoz 1983 : complex p-adic dynamical systems,
- Volovich 1987 : p-adic string theory.

We are interested in the **polynomials** and **rational maps** considered as dynamical systems on $\mathbb{Z}_p$, $\mathbb{Q}_p$ or $\mathbb{P}^1(\mathbb{Q}_p)$.

As first investigations, we consider two families of dynamical systems :

1-**Lipschitz** dynamical systems and **expanding** dynamical systems.

- 1-Lipschitz means $|f(x) - f(y)|_p \leq |x - y|_p$.
- expanding means $|f(x) - f(y)|_p > |x - y|_p$.

III. Typical polynomial dynamical systems on $\mathbb{Z}_p$

The 1-Lipschitz dynamics $f(x) = x + 1$ on $\mathbb{Z}_p$ is minimal.

- f is called **minimal**, if every orbit $\{f^n(x) : n \in \mathbb{N}\}$ is dense.
- Recall : $\overline{\mathbb{N}} = \mathbb{Z}_p$.

The expanding dynamics $f(x) = \frac{x^p - x}{p}$ on $\mathbb{Z}_p$ is conjugate to the shift on $\Sigma_p := \{0, 1, \ldots, p-1\}^{\mathbb{N}}$.

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{\ f\ } & \mathbb{Z}_p \\ \phi\downarrow & & \downarrow\phi \\ \Sigma_p & \xrightarrow[\ \sigma\ ]{} & \Sigma_p \end{array}$$

- Remark : shift σ is defined as

$$x = x_0 x_1 x_2 \cdots \stackrel{\sigma}{\longmapsto} \sigma x = x_1 x_2 \cdots .$$

1-Lipschitz dynamical systems on $\mathbb{Q}_p$

I. 1-Lipschitz p-adic dynamical systems

Let $f \in \mathbb{Z}_p[x]$ be a polynomial of coefficients in $\mathbb{Z}_p$. Then it defines a dynamical system on $\mathbb{Z}_p$, denoted by $(\mathbb{Z}_p, f)$.

It is 1-**Lipschitz** and then **equicontinuous**.

The system (X, T) is **equicontinuous** if

$$\forall \epsilon > 0, \exists \delta > 0 \ \text{ s. t. } \ d(T^n x, T^n y) < \epsilon \ (\forall n \geq 1, \forall d(x, y) < \delta).$$

Theorem

Let X be a compact metric space and $T : X \to X$ be an *equicontinuous transformation*. Then the following statements are equivalent :
(1) T is **minimal** (every orbit is dense).
(2) T is **ergodic** with respect to the Haar measure.

II. 1-Lipschitz continuous dynamics on $\mathbb{Z}_p$

For 1-Lipschitz continuous maps $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, the dynamical systems $(\mathbb{Z}_p, f)$ are extensively studied. For example :

- Polynomials :
  Coelho-Parry 2001 : ax and distribution of Fibonacci numbers
  Gundlach-Khrennikov-Lindahl 2001 : ergodicity of x^n on spheres.
  A. Fan-Li-Yao-Zhou 2007 : minimal decomposition of $ax + b$.
  Durand-Paccaut 2009 : minimal polynomials on $\mathbb{Z}_3$.
  Diarra-Sylla 2014 : periodic orbits of Chebyshev polynomials.
  Memić 2017 : ergodic polynomials on 2-adic spheres.
  $\rightarrow$ More on the next slide.
- Mahler Series
  Anashin 1994, 1995, 1998, 2002 ; Jeong 2015, 2018 ; Jeong-Li 2017 ; Memić 2020....
- van der Put Series
  Yurova 2010 ; Anashin-Khrennikov-Yurova 2011, 2012, 2014 ; Khrennikov-Yurova 2011 ; Jeong 2013, 2018....
- T-functions
  Anashin-Khrennikov-Yurova 2014 ; Wang-Hu-Liu 2018....

III. Polynomial dynamical systems on $\mathbb{Z}_p$

Theorem (Ai-Hua Fan, L ; 2011) minimal decomposition

Let $f \in \mathbb{Z}_p[x]$ with $\deg f \geq 2$. The space $\mathbb{Z}_p$ can be decomposed into :

$$\mathbb{Z}_p = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B},$$

where

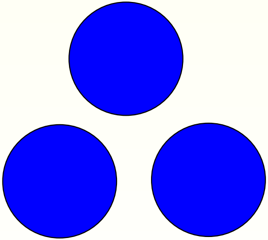
- $\mathcal{P}$ is the finite set consisting of all periodic orbits ;
- $\mathcal{M} := \sqcup_{i \in I} \mathcal{M}_i$ (I finite or countable)
  $\rightarrow$ $\mathcal{M}_i$: finite union of balls, $f : \mathcal{M}_i \to \mathcal{M}_i$ is minimal ;
- $\mathcal{B}$ is attracted into $\mathcal{P} \sqcup \mathcal{M}$.

Remark : Similar decomposition theorem holds for

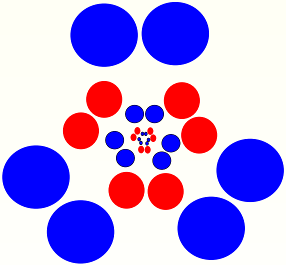
- linear fractional maps on $\mathbb{P}^1(\mathbb{Q}_p)$ (A.Fan–S.Fan–**L**–Wang 2014),
- power series in a finite extension of $\mathbb{Q}_p$ (S.Fan–**L** 2015),
- rational maps having good reduction on $\mathbb{P}^1(\mathbb{Q}_p)$ (A.Fan–S.Fan–**L**–Wang 2017).

More results : S.Fan-**L** 2016a, 2016b ; Jung-Kim-Song 2019 ; Jung-Kim 2020 ; Kim-Kwon-Song 2020....

Two typical decompositions of $\mathbb{Z}_p$



$Tx = x + 3, \; p = 3$



$Tx = 6x, \; p = 7$

IV. One application in Number Theory

Proposition (Fan-Li-Yao-Zhou 2007)

Let $k \geqslant 1$ be an integer, and let a, b, c be three integers in $\mathbb{Z}$ coprime with $p \geqslant 2$. Let s_k be the least integer $\geqslant 1$ such that $a^{s_k} \equiv 1 \pmod{p^k}$.

(a) If $b \not\equiv a^j c \pmod{p^k}$ for all integers j $(0 \leqslant j < s_k)$, then $p^k \nmid (a^n c - b)$, for any integer $n \geqslant 0$.

(b) If $b \equiv a^j c \pmod{p^k}$ for some integer j $(0 \leqslant j < s_k)$, then we have

$$\lim_{N \to +\infty} \frac{1}{N} \mathrm{Card}\{1 \leqslant n < N : p^k \mid (a^n c - b)\} = \frac{1}{s_k}.$$

Remark : Consider $T : x \mapsto ax$. Then

$$p^k \mid (a^n c - b) \Leftrightarrow |T^n(c) - b|_p \leq p^{-k} \Leftrightarrow T^n(c) \in \overline{B}(b, p^{-k}).$$

Coelho and Parry 2001 : Ergodicity of p-adic multiplications and the distribution of Fibonacci numbers.

Expanding dynamical systems on $\mathbb{Q}_p$

I. Expanding dynamical systems on $\mathbb{Q}_p$

- $f\colon X(\subset \mathbb{Q}_p) \to \mathbb{Q}_p$ is **continuously differentiable** at $a \in X$ if

$$\lim_{(x,y)\to(a,a),x\neq y} \frac{f(x)-f(y)}{x-y} =: f'(a) \quad \text{exists.}$$

Lemma (Local rigidity lemma)

Let U be a clopen (close and open) set and $a \in U$. Suppose $f\colon U \to \mathbb{Q}_p$ is continuously differentiable, and $f'(a) \neq 0$. Then there exists $r > 0$ such that $B_r(a) \subset U$ and

$$\forall x, y \in B_r(a), \quad |f(x)-f(y)|_p = |f'(a)|_p |x-y|_p.$$

II. Some results on expanding dynamics in $\mathbb{Q}_p$

Thiran–Verstegen–Weyers 1989 : quadratic maps.

Woodcock–Smart 1998 : $(x^p - x)/p$.

Shabat 2004 : logistic maps : $\lambda x(1-x)$.

Dremov–Shabat–Vytnova 2006 : quadratic maps.

Mukhamedov–Mendes 2007 : generalized logistic maps : $(ax)^2(x+1)$.

Fan-**L**-Wang-Zhou 2007 : p-adic repeller.

Kingsbery-Levin-Preygel-Silva 2009 : $\binom{x}{p^\ell}$ on $\mathbb{Z}_p$.

Zelenov 2014 : p-adic baker's transformation.

Furno 2016 : Hausdorff dimension of p-adic Julia sets.

Mukhamedov-Khakimov 2017, 2018 ; Ahmad–**L**–Saburov 2018 :
Potts-Bethe mapping $\left(\frac{\theta x+q-1}{x+\theta+q-2}\right)^k$.

Fan-**L** 2018a, 2018b : $ax + \frac{1}{x}$ on $\mathbb{P}^1(\mathbb{Q}_p)$.

III. p-adic repeller

- $f : X \to \mathbb{Q}_p$, $X \subset \mathbb{Q}_p$ compact open.
- Assume that
  1. $f^{-1}(X) \subset X$;
  2. $X = \bigsqcup_{i\in I} B_{p^{-\tau}}(c_i)$ (with some $\tau \in \mathbb{Z}$), $\forall i \in I$, $\exists \tau_i \in \mathbb{Z}$ s.t.

$$|f(x) - f(y)|_p = p^{\tau_i}|x - y|_p \qquad (\forall x, y \in B_{p^{-\tau}}(c_i)). \tag{1}$$

- Define **Filled Julia set** :

$$J_f = \bigcap_{n=0}^{\infty} f^{-n}(X).$$

We have $f(J_f) \subset J_f$. (X, J_f, f) is called

$\to$ a **p-adic weak repeller** if all $\tau_i \geq 0$ in (1), but at least one > 0.

$\to$ a **p-adic repeller** if all $\tau_i > 0$ in (1).

IV. Description by subshift of finite type

- For any $i \in I$, let

$$I_i := \{j \in I : B_j \cap f(B_i) \neq \emptyset\} = \{j \in I : B_j \subset f(B_i)\}.$$

- Define $A = (A_{i,j})_{I \times I}$:

$$A_{ij} = 1 \text{ if } j \in I_i; \quad A_{ij} = 0 \text{ otherwise.}$$

- If A is irreducible, we say that (X, J_f, f) is **transitive**.
- Let (Σ_A, σ) be the corresponding subshift.

Theorem (A.Fan–L–Wang–Zhou, 2007)

Let (X, J_f, f) be a transitive p-adic weak repeller with matrix A. Then the dynamics (J_f, f) is topologically conjugate to the shift dynamics (Σ_A, σ).

V. Examples and applications

Let $a \in \mathbb{Z}_p,\ a \equiv 1 \pmod{p}$ and $m \geq 1$ be an integer.
Consider $f_{m,a} : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p$:

$$f_{m,a}(x) = \frac{x^p - ax}{p^m}.$$

- $I_{m,a} = \{0 \leq k < p^m : k^p - ak \equiv 0 \pmod{p^m}\}$
- $X_{m,a} = \bigsqcup_{k \in I_{m,a}} (k + p^m \mathbb{Z}_p), \qquad J = \bigcap_{n=0}^{\infty} f_m^{-n}(X).$

Theorem (A.Fan–L–Wang–Zhou, 2007)

$(J, f_{m,a})$ is conjugate to $(\{0, \dots, p-1\}^{\mathbb{N}}, \sigma)$.

Theorem (Woodcock and Smart 1998)

$(J, f_{1,1})$ is conjugate to $(\{0, \dots, p-1\}^{\mathbb{N}}, \sigma)$.

Applications : existence of periodic p-adic Gibbs measures on Cayley trees (Mukhamedov 2012, 2013 ; Akin-Dogan-Mukhamedov 2017 ; Mukhamedov-Khakimov 2017 ; Ahmad–**L**–Saburov 2018...)

Geometrically finite rational dynamics

I. Some studies rational maps on $\mathbb{Q}_p$

Mukhamedov–Rozikov 2004 : $\frac{x+a}{bx+c}$.

Khamraev–Mukhamedov 2006 : $\frac{ax^2}{bx+1}$.

Dragovich–Khrennikov–Mihajlović 2007 : rational maps of degree 1 on the adelic space.

Diao–Silva 2011 : 1-Lipschitz properties of rational maps.

Albeverio–Rozikov–Sattarov 2013 : $(2,1)$-rational maps on $\mathbb{C}_p$.

Sattarov 2015 : $(3,2)$-rational maps on $\mathbb{C}_p$.

Rozikov-Sattarov 2017 : $(2,2)$-rational maps with unique fixed point.

Rozikov-Sattarov-Yam 2019 : $\frac{ax}{x^2+a}$.

Rozikov-Sattarov 2020 : $(2,2)$-rational maps with two fixed points.

II. Fatou set and Julia set

Let (X, f) be a dynamical system.

- **Fatou set** is the set of points in X having a neighborhood on which $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.
- **Julia set** $J_X(f)$ is the complement of of $F_X(f)$.

Remark : Usually,

- on its Fatou set f is 1-Lipschitz and the dynamics is described by minimal decomposition,
- on its Julia set f is expanding and the dynamics is described by some Markov shift.

III. Dynamics on the Julia set

We study the dynamics $(\mathbb{P}^1_K, f)$ where

- K is a finite extension of $\mathbb{Q}_p$, and $\mathbb{P}^1_K$ is its projective line
- $f \in K(z)$ is a rational map.

- $\mathrm{Crit}_J(f)$: set of the critical points (x such that $f'(x) = 0$) in $J_K(f)$.
- Grand orbit of the critical points in $\mathrm{Crit}_J(f)$:

$$\mathrm{GO}(\mathrm{Crit}_J(f)) := \{y \in K : \exists x \in \mathrm{Crit}_J(f), \exists m, n \in \mathbb{N}, s.t.\ f^n(y) = f^m(x)\}.$$

- $I_K(f) := J_K(f) \setminus \mathrm{GO}(\mathrm{Crit}_J(f))$.

The map $f \in K(z)$ is *geometrically finite* if every critical point in $\mathrm{Crit}_J(f)$ has finite forward orbit.

Theorem (S.Fan–L–Nie–Wang, in preparation)

Let $f \in K(z)$ be a geometrically finite rational map with $\deg(f) \geq 2$. Then there exist a countable states Markov shift (Σ_A, σ) and a bijection $h : I_K(f) \to \Sigma_A$ s.t. $(I_K(f), f)$ is topologically conjugate to (Σ_A, σ) via h.

An example :
$x \mapsto \frac{9}{4}x(x-1)^2$ on $\mathbb{P}^1(\mathbb{Q}_2)$

I. Fixed points and critical orbits

Consider $f : \mathbb{P}^1(\mathbb{Q}_2) \to \mathbb{P}^1(\mathbb{Q}_2)$ given by

$$f(x) = \frac{9}{4}x(x-1)^2.$$

- Fixed points : 0, $1/3$ and ∞.
- Critical points : 1, $1/3$ and ∞.
- $1/3$ and ∞ are the super-attracting.
- 0 is the repelling fixed point with multiplier 9/4.

We have the following critical portrait of f :

$$1 \xrightarrow{2:1} 0 \circlearrowleft 1:1 \qquad \frac{1}{3} \circlearrowleft 2:1 \qquad \infty \circlearrowleft 3:1$$

Remark : the map f is the unique cubic postcritically finite polynomial with rational coefficients such that the corresponding p-adic Julia set contains a critical point.

II. Fatou set and Julia set of the map $x \mapsto \frac{9}{4}x(x-1)^2$

Denote by $F(f)$ and $J(f)$ the Fatou set and the Julia set of f.

For $a \in F(f)$, denote by Ω_a the Fatou component of f containing a.

Proposition

The Julia set of f is

$$J(f) = \bigcap_{n \geq 1} f^{-n}(4\mathbb{Z}_2 \cup 1 + 4\mathbb{Z}_2).$$

Collorary

The Fatou set of f is

$$F(f) = \bigcup_{n \geq 0} f^{-n}(\Omega_\infty \cup \Omega_{1/3}).$$

III. Basic facts of the map $x \mapsto \frac{9}{4}x(x-1)^2$

We have

- $\mathbb{P}^1(\mathbb{Q}_2) \setminus \mathbb{Z}_2 \subset \Omega_\infty$.
- $(2 + 4\mathbb{Z}_2) \cup (1 + 2 + 4\mathbb{Z}_2) \subset F(f)$.
    - $1 + 2 + 4\mathbb{Z}_2 \subset \Omega_{1/3}$.
    - $f(2 + 4\mathbb{Z}_2) \subset \Omega_\infty$.
- f is a scaling on $4\mathbb{Z}_2$ with scaling ratio 4. Moreover, for $n \geq 1$ and $a, b \in \{0, 1\}$,
    - $f(2^{n+1}\mathbb{Z}_2) = 2^{n-1}\mathbb{Z}_2$.
    - $f(2^{n+1} + 2^{n+2}\mathbb{Z}_2) = 2^{n-1} + 2^n\mathbb{Z}_2$.
    - $f(2^{n+1} + a2^{n+2} + 2^{n+3}\mathbb{Z}_2) = 2^{n-1} + a2^n + 2^{n+1}\mathbb{Z}_2$.
    - $f(2^{n+1} + a2^{n+2} + b2^{n+3} + 2^{n+4}\mathbb{Z}_2) = 2^{n-1} + a2^n + b2^{n+1} + 2^{n+2}\mathbb{Z}_2$.

IV. Basic facts of the map $x \mapsto \frac{9}{4}x(x-1)^2$, continued

For $n \geq 1$,

- $f(2^{2n} + 2^{2n+3}\mathbb{Z}_2) = 2^{2n-2} + 2^{2n+1}\mathbb{Z}_2$. In particular,
$$f(2^2+2^5\mathbb{Z}_2) = \{1\} \cup \bigcup_{m\geq 2} (1+2^{m+1}+2^{m+3}\mathbb{Z}_2 \cup 1+2^{m+1}+2^{m+2}+2^{m+3}\mathbb{Z}_2).$$
- $f(2^{2n} + 2^{2n+2} + 2^{2n+3}\mathbb{Z}_2) = 2^{2n-2} + 2^{2n} + 2^{2n+1}\mathbb{Z}_2$.
- $f(2^{2n+1} + 2^{2n+2}\mathbb{Z}_2) = 2^{2n-1} + 2^{2n}\mathbb{Z}_2$. In particular,
$$f^{n+1}(2^{2n-1} + 2^{2n}\mathbb{Z}_2) \subset \Omega_\infty.$$
- $f(2^{2n} + 2^{2n+1} + 2^{2n+2}\mathbb{Z}_2) = 2^{2n-2} + 2^{2n-1} + 2^{2n}\mathbb{Z}_2$. In particular,
$$f^n(2^{2n} + 2^{2n+1} + 2^{2n+2}\mathbb{Z}_2) = \Omega_{1/3}.$$
- $f(1 + 2^n\mathbb{Z}_2) \subset 2^{2n-2}\mathbb{Z}_2$.
- $f(1 + 2^{n+1} + 2^{n+3}\mathbb{Z}_2) = 2^{2n} + 2^{2n+3}\mathbb{Z}_2$. In particular,
f is a scaling on $1 + 2^{n+1} + 2^{n+3}\mathbb{Z}_2$ with scaling ratio 2^{-n}.
- $f(1 + 2^{n+1} + 2^{n+2} + 2^{n+3}\mathbb{Z}_2) = 2^{2n} + 2^{2n+3}\mathbb{Z}_2$. In particular,
f is a scaling on $1 + 2^{n+1} + 2^{n+2} + 2^{n+3}\mathbb{Z}_2$ with scaling ratio 2^{-n}.

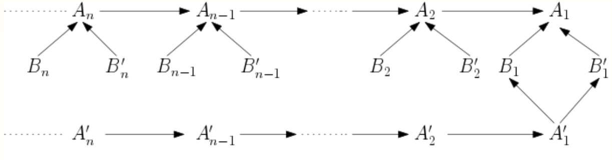
V. Dynamics of $x \mapsto \frac{9}{4}x(x-1)^2$

For $n \geq 1$, define

$$A_n := 2^{2n} + 2^{2n+3}\mathbb{Z}_2, \quad A'_n := 2^{2n} + 2^{2n+2} + 2^{2n+3}\mathbb{Z}_2,$$

$$B_n := 1 + 2^{n+1} + 2^{n+3}\mathbb{Z}_2, \quad B'_n := 1 + 2^{n+1} + 2^{n+2} + 2^{n+3}\mathbb{Z}_2.$$

Then under f, we have the following portrait :



Recall : $A_1 = 2^2 + 2^5\mathbb{Z}_2$ and

$$f(2^2 + 2^5\mathbb{Z}_2) = \{1\} \cup \bigcup_{m\geq 2} (1 + 2^{m+1} + 2^{m+3}\mathbb{Z}_2 \cup 1 + 2^{m+1} + 2^{m+2} + 2^{m+3}\mathbb{Z}_2)$$

$$= \{1\} \cup \bigcup_{m\geq 2} (B_m \cup B'_m).$$

VI. Countable states Markov shift - symbols

Set

$$\alpha_n := A_n \cap J(f), \ \alpha'_n := A'_n \cap J(f), \ \beta_n := B_n \cap J(f), \ \beta'_n := B'_n \cap J(f),$$

and define

$$\alpha_\infty := \{0\} \quad \text{and} \quad \beta_\infty := \{1\}.$$

Denote

$$\mathcal{A} := \{\alpha_\infty, \beta_\infty, \alpha_1, \alpha'_1, \beta_1, \beta'_1, \cdots, \alpha_n, \alpha'_n, \beta_n, \beta'_n, \cdots\}.$$

Let the matrix $A = (A_{\gamma_i,\gamma_j})_{\gamma_i,\gamma_j \in \mathcal{A}}$ be given by

$$A_{\gamma_i,\gamma_j} = 1 \text{ if } \gamma_j \subset f(\gamma_i), \text{ and } A_{\gamma_i,\gamma_j} = 0 \text{ otherwise.}$$

Recall :

$$f(\alpha_1) = \beta_\infty \cup \bigcup_{n \geq 2} \{\beta_n, \beta'_n\}.$$

and

$$f(\beta_\infty) = \alpha_\infty, \quad \text{and} \quad f(\alpha_\infty) = \alpha_\infty.$$

VII. Countable states Markov shift - the matrix

The matrix A has a unique irreducible component A'. It corresponds to the symbol set $\mathcal{A}' = \{\alpha_1, \alpha_2, \beta_2, \beta_2', \cdots, \alpha_n, \beta_n, \beta_n', \cdots\}$.

$\rightarrow$ The corresponding subsystem $(\Sigma'_{A'}, \sigma')$ has the following matrix

	α_1	α_2	β_2	β_2'	α_3	β_3	β_3'	α_4	β_4	β_4'	α_5	β_5	β_5'	$\dots$
α_1			1	1		1	1		1	1		1	1	$\dots$
α_2	1													$\dots$
β_2		1												$\dots$
β_2'		1												$\dots$
α_3		1												$\dots$
β_3					1									$\dots$
β_3'					1									$\dots$
α_4					1									$\dots$
β_4								1						$\dots$
β_4'								1						$\dots$
α_5								1						$\dots$
β_5											1			$\dots$
β_5'											1			$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

VIII. The Gurevich entropy

Gurevich entropy : $h_G(\Sigma_A, \sigma) = \sup h_{\text{top}}(\Sigma'_A, \sigma)$, where the supremum is over all subsystems (Σ'_A, σ) formed by restricting to a finite subset of symbols.

Suppose A is irreducible. For any $\gamma_{i_0} \in \Sigma$, a first-return loop of length $n \geq 1$ at γ_{i_0} is a path $\{\gamma_{i_0}, \gamma_{i_1}, \cdots, \gamma_{i_n}\}$ such that

$$\gamma_{i_0} = \gamma_{i_n},\ \gamma_{i_k} \neq \gamma_{i_0} (1 \leq k \leq n-1), \&\ A_{\gamma_{i_k}, \gamma_{i_{k+1}}} = 1, (0 \leq k \leq n-1).$$

- $\delta_{\gamma_{i_0}}(n)$: number of first return loops at γ_{i_0} of length n, and set
- $S_{\gamma_{i_0}}(z) = \sum_{n=1}^{\infty} \delta_{\gamma_{i_0}}(n) z^n.$

Proposition : If $1 - S_{\gamma_{i_0}}(z)$ has a real root $R > 0$ such that $S_{\gamma_{i_0}}(z)$ converges and is not 1 on $|z| < R$, then $h_G(\Sigma_A, \sigma) = -\log R$.
(If such R exists, then it is independent of γ.)

In our case, $\delta_{\alpha_1} = 0$ if $n \leq 2$ and $\delta_{\alpha_1} = 2$ if $n \geq 3$, thus

$$S_{\alpha_1}(z) = \sum_{n=3}^{\infty} 2z^n = \frac{2z^3}{1-z}, \quad h_G(\Sigma'_{A'}, \sigma') = -\log R \approx 0.528057....$$